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Krein space quantization in curved and flat spacetimes

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Abstract

We re-examine in detail a canonical quantization method *à la* Gupta–Bleuler in which the Fock space is built over a so-called Krein space. This method has already been successfully applied to the massless minimally coupled scalar field in de Sitter spacetime for which it preserves covariance. Here, it is formulated in a more general context. An interesting feature of the theory is that, although the field is obtained by canonical quantization, it is independent of Bogoliubov transformations. Moreover, no infinite term appears in the computation of $T^{\mu\nu}$ mean values and the vacuum energy of the free field vanishes: $\langle 0 | T^{00} | 0 \rangle = 0$. We also investigate the behaviour of the Krein quantization in Minkowski space for a theory with interaction. We show that one can recover the usual theory with the exception that the vacuum energy of the free theory is zero.

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1. Introduction

Recently a covariant quantization of the massless minimally coupled scalar field on de Sitter (dS) spacetime has been carried out [1]. Indeed, the quantization scheme used is not restricted to the dS spacetime; in this paper we describe it in a more general context.

Although quantum field theory (QFT) on a curved classical background has already been extensively studied [2, 3], various problems are still open. This is even true for the de Sitter space, which is of current importance. The dS space really appears as one of the simplest curved spacetimes: it is maximally symmetric and offers the opportunity of controlling the transition to the flat spacetime by the so-called contraction procedure [4]. In view of these properties dS spacetime should at least be considered as an excellent laboratory. Nevertheless, even for this very simple case, one encounters difficulties in defining quantum fields, including the free fields.

First, the one-particle state space is usually taken as the carrier space of a unitary irreducible representation of the symmetry group (elementary system in Wigner's sense). In the massive Minkowskian case, the carrier space is a Hilbert space which can be selected on physical grounds by the condition of energy positiveness. But since the energy concept is defined only for stationary spacetimes, it turns out that, in general, one cannot characterize in a unique way a preferred invariant subspace of solutions of the field equation. This kind of difficulty actually corresponds to the choice of a vacuum state among a family of the so-called α -vacua introduced in [5–7]. The choice of a vacuum among the family of α -vacua has been recently largely discussed in various contexts related to inflation [8, 9], dS/CFT correspondence [10–13] or QFT [14–16, 4] and no conclusive arguments which would singled out a preferred vacuum state have been given so far.

But it becomes worse. A famous example is given by Allen's no-go theorem concerning the so-called massless minimally coupled scalar field in dS space for which it has been claimed that no invariant vacuum exists. That is to say that no covariant Hilbert space quantization is possible [5]. It actually happens that no invariant Hilbert space \mathcal{H}_p containing every regular solutions of the field equation, with initial conditions having a compact support, can be defined at all. This difficulty is still present for the dS massless spin-2 field.

In this paper, we present a new version of the canonical quantization which allows covariant quantization even in situations where the usual method fails, including the dS massless minimally coupled field and spin-2 massless field. This construction is of the Gupta–Bleuler type and the set of states is different from the set of physical states. Instead of having a multiplicity of vacua, we have several possibilities for the space of physical states and only one field and one vacuum, the latter being invariant under Bogoliubov transformations. So the usual ambiguity about vacua is not suppressed but displaced. Our construction gives a framework (the Krein space) where all objects such as field, Bogoliubov transformations and vacuum can be treated together in a very straightforward way (for an alternative approach, see [17–19]). Moreover this field presents an interesting property linked to the cosmological constant problem: the vacuum energy of the free field vanishes without any reordering or regularization.

We also investigate how this method behaves in the Minkowski space when interaction is present. In fact, a successful quantization scheme in the dS space must lead to the common results when applied in the usual flat case.

Section 2 gives a brief overview of the problem of quantizing a free scalar field in a fixed curved spacetime background. In section 3, we introduce a simple and basic independent rigorous formulation in which the minimal requirements for a canonical quantization are formulated. We explain in section 4 a new method based on weaker conditions, which do not prohibit negative norm states in the definition of the field. Actually the quantization will be done in the larger framework of a Krein space which will preserve the covariance. Section 5 is devoted to the flat case with a self-coupling.

2. Free fields in a curved spacetime

Let us consider a spacetime M , that is a pseudo-Riemannian manifold endowed with a metric $g_{\mu\nu}(x)$ with a signature $(+, -, -, -)$. We restrict ourselves to the free scalar field whose Lagrangian density reads

$$\mathcal{L} = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \zeta^2 \phi,$$

where ζ is a real coupling parameter whose meaning will not be discussed here. The

corresponding field equation reads

$$\square \phi + \zeta^2 \phi = 0. \quad (1)$$

The spacetime is supposed to be globally hyperbolic [20, 21]. Thus, the field equation admits Cauchy space-like surfaces Σ and a two-point function \tilde{G} which satisfies

$$\phi(x) = \int_{\Sigma} \tilde{G}(x, x') \overleftrightarrow{\partial}_{\mu} \phi(x') d\sigma^{\mu}(x'), \quad (2)$$

for each regular solution ϕ . Although it is a purely geometrical object, the function \tilde{G} is referred to as the commutator of the field. This function \tilde{G} is a solution of the field equation in each of its variables and, for a given x , its support in x' belongs to the causal cone of x . Hence, the classical field ϕ is determined by its initial conditions on the surface Σ . Consistently with (2) the space of solutions of the field equation is endowed with the Klein–Gordon scalar product,

$$\langle \phi_1, \phi_2 \rangle = i \int_{\Sigma} \phi_1^* \overleftrightarrow{\partial}_{\mu} \phi_2 d\sigma^{\mu}, \quad (3)$$

which is independent of the Cauchy surface Σ . Note that (3) may be indefinite. In flat spacetime, this scalar product is nothing but the natural scalar product for the functions defined on the mass hyperboloid in the momentum space.

The usual (L^2) scalar product

$$(f_1, f_2) = \int_M f_1^*(x) f_2(x) d\lambda(x), \quad (4)$$

where $d\lambda$ is the natural measure $d\lambda(x) = \sqrt{|g(x)|} d^4x$, is also defined for functions on the manifold. Contrary to the Klein–Gordon product, this product is positive definite and the functions f_1 and f_2 are not necessarily solutions of the field equation.

Some very important spacetimes, Minkowski of course but also de Sitter and anti de Sitter spacetimes, admit a large isometry group G . This group acts in a natural way on the solutions through $U_g \phi(x) = \phi(g^{-1}x)$ for $g \in G$. The Klein–Gordon and the L^2 products are left invariant under this action, thus one has

$$\langle U_g \phi_1, U_g \phi_2 \rangle = \langle \phi_1, \phi_2 \rangle,$$

and

$$(U_g f_1, U_g f_2) = (f_1, f_2).$$

Quantizing such a field amounts to building an operator-valued distribution φ , rigorously defined by $\varphi(f)$, where f is a test function, i.e., a C^{∞} function with compact support on M . For convenience, we will use the common abusive notation $\varphi(x)$. The field must satisfy minimal requirements to be acceptable: it must verify the field equation, commute for space-like separations (micro-causality) and satisfy the covariance condition

$$\underline{U}_g^{-1} \varphi(x) \underline{U}_g = \varphi(g \cdot x), \quad \forall x \in M, \quad \forall g \in G.$$

It is well known that the quantization procedure based on the Hilbert space structure and satisfying the above conditions already fails in many situations. For instance, in QED, the only way to preserve (manifest) covariance and gauge invariance in canonical quantization is to use the Gupta–Bleuler method. Similarly, it has been shown [5] that, for the minimally coupled scalar field ($\zeta = 0$ in (1)) in de Sitter spacetime, no Hilbert space can be defined for the quantization. The price to pay in both cases in building a covariant quantum field is the appearance of unphysical negative norm states. It is commonly accepted that the very reason for that impossibility relies, in the case of electrodynamics, upon the invariance of

the Lagrangian under a gauge transformation. Actually this also occurs for the dS massless minimally coupled scalar field, where the transformation ‘ $\phi \rightarrow \phi + \text{constant}$ ’, viewed as a gauge transformation, is a symmetry of the Lagrangian.

In this paper we present a method which within the framework of canonical quantization generalizes the Gupta–Bleuler procedure to situations in which the occurrence of negative norm states is unavoidable in order to achieve a covariant quantization. More precisely, as for the Gupta–Bleuler method, the field will be written in terms of a Krein space and the task will consist of finding the conditions which define the one-particle sector.

3. Canonical quantization

Before presenting the Krein space method, let us indicate a rigorous and intrinsic formulation of the usual canonical quantization. This material will be used for the Krein space quantization as well. Actually, we will see that our method is also a canonical quantization which only differs from the usual one by the representation of the canonical commutation relations.

We first give a heuristic overview of the canonical quantization method, which can be summarized as follows. Let ϕ_k be a family of solutions (supposed to be known) of the field equation and verifying

$$\langle \phi_k, \phi_{k'} \rangle = \delta_{kk'}, \quad (5)$$

$$\langle \phi_k, \phi_{k'}^* \rangle = 0, \quad (6)$$

$$\sum_k \phi_k(x) \phi_k^*(x') - \phi_k^*(x) \phi_k(x') = -i\tilde{G}(x, x'). \quad (7)$$

The choice of the modes (ϕ_k) determines the space \mathcal{H}_p they generate as a Hilbertian basis. That space is usually called the one-particle sector, although the word ‘particle’ may be confusing. The question of how to make this choice is answered for stationary spacetimes (Minkowski) by the fact that one can find a Killing vector field X which is time-like at each point, and thus naturally defines a Hamiltonian for the quantization space: iX . The space \mathcal{H}_p is then chosen in such a way that the Hamiltonian admits a positive spectrum; the modes are generally chosen as eigenvectors, with positive eigenvalues, of that operator. The corresponding modes ϕ_k are said to be positive frequency modes.

Whenever the considered spacetimes are non-stationary, one is left with the orthonormality conditions alone to characterize \mathcal{H}_p . Unfortunately, these do not uniquely determine the one-particle space. Actually, if the ϕ_k satisfy the above conditions so do the family $\tilde{\phi}_k = A\phi_k - B\phi_k^*$ with $|A|^2 - |B|^2 = 1$ (Bogoliubov transformation). The question of which family should be preferred is not trivial since the two-point functions (thus also the expectation values) will depend on that choice. Because the vacuum is defined through the modes, it has become usual to say that the choice of the modes is equivalent to the choice of the vacuum. We will see that our point of view is different.

The third condition (7) ensures the completeness of the family $\{\phi_k\}$. The space spanned by the ϕ_k and the ϕ_k^* contains all the regular solutions of the field equation for which the initial conditions have a compact support.

The quantum field is now defined through

$$\varphi(x) = \sum_k \phi_k(x) A_k + \phi_k^*(x) A_k^\dagger, \quad (8)$$

where A_k and A_k^\dagger are operators verifying the canonical commutation relations (ccr)

$$[A_k, A_{k'}^\dagger] = \delta_{kk'}, \quad [A_k, A_{k'}] = 0, \quad [A_k^\dagger, A_{k'}^\dagger] = 0.$$

Our notation departs from the usual one (with small letters) because the above relations are the starting point of both the usual and what we shall call in the following the Krein canonical quantization. These two quantizations will differ only through the representation of these relations, that is through the realization of the A_k as operators acting on some vector space. Before any representation has been set, it is clear that, at least formally, such a field satisfies the field equation and the causality requirement since the ccr imply

$$[\varphi(x), \varphi(x')] = -i\tilde{G}(x, x').$$

The covariance condition seems not so clear but we shall use a formalism which makes it obvious, the only requirement being that the space \mathcal{H}_p is closed under the action of the isometry group. Recall that the space \mathcal{H}_p is defined through the square-summability requirement

$$\mathcal{H}_p = \left\{ \sum_k c_k \phi_k; \sum_k |c_k|^2 < \infty \right\}.$$

Regarding the usual canonical quantization, the representation of the ccr is realized in the Fock space $\underline{\mathcal{H}}_p$ (see the appendix) by

$$A_k = a_k := a(\phi_k) \quad \text{and} \quad A_k^\dagger = a_k^\dagger := a^\dagger(\phi_k).$$

It follows from (8) and the linearity and anti-linearity of the creation and annihilation operators respectively that

$$\varphi(x) = a(p(x)) + a^\dagger(p(x)),$$

where $p(x)$ is defined through

$$p(x) = \sum_k \phi_k^*(x) \phi_k.$$

The key point is that p can be defined without reference to the basis. For that recall that p is a \mathcal{H}_p -valued distribution. Consequently, it must be smeared by test functions (which are chosen to be real) $f \in \mathcal{C}_o^\infty(M)$. The rigorous definition of φ reads

$$\begin{aligned} \varphi(f) &= \int f(x) \varphi(x) \, d\mu(x) \\ &= \sum_k (\phi_k^*, f) a_k + \sum_k (\phi_k, f) a_k^\dagger, \end{aligned}$$

where the parenthesis stands for the L^2 product. Using the linearity one obtains

$$\varphi(f) = a \left(\sum_k (\phi_k, f) \phi_k \right) + a^\dagger \left(\sum_k (\phi_k, f) \phi_k \right).$$

Defining

$$p(f) = \sum_k (\phi_k, f) \phi_k \in \mathcal{H}_p, \quad (9)$$

one finally gets

$$\varphi(f) = a(p(f)) + a^\dagger(p(f)), \quad (10)$$

where p is a \mathcal{H}_p -valued distribution. A base-free definition of p is that $p(f)$ is the only element of the Hilbert space \mathcal{H}_p for which

$$\langle p(f), \psi \rangle = (f, \psi), \quad \forall \psi \in \mathcal{H}_p, \quad (11)$$

where the scalar products $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) are respectively defined in equations (3) and (4). Expressions (10) and (11) then allow us to give a rigorous definition of the field which depends on the choice of \mathcal{H}_p but not on the basis given by the modes. The covariance of the distribution p easily follows from this definition under the sole hypothesis that \mathcal{H}_p is closed with respect to the action of the group G . More precisely, we show that p intertwines the natural representations U of G over $\mathcal{C}_0^\infty(M)$ and \mathcal{H}_p :

$$U_g p = p U_g. \quad (12)$$

Indeed, for $\psi \in \mathcal{H}_p$ one has

$$\begin{aligned} \langle U_g p(f), \psi \rangle &= \langle p(f), U_g^{-1} \psi \rangle \\ &= (f, U_g^{-1} \psi) \\ &= (U_g f, \psi) \\ &= \langle p(U_g f), \psi \rangle. \end{aligned}$$

The representation U is extended to a representation \underline{U} on $\underline{\mathcal{H}}_p$, and therefore

$$\begin{aligned} \underline{U}_g \varphi(f) \underline{U}_g^{-1} &= a(U_g p(f)) + a^\dagger(U_g p(f)) \\ &= a(p(U_g f)) + a^\dagger(p(U_g f)) \\ &= \varphi(U_g f). \end{aligned}$$

Thus, the usual canonical quantization, and its covariance, rests on the existence of a space of solution \mathcal{H}_p in which the scalar product is positive, which is closed under the group action and such that $\mathcal{H}_p + \mathcal{H}_p^*$ contains all the regular solution of the equation. It is important to mention a technical point: in definition (11) it is assumed that the map $\psi \mapsto (f, \psi)$ is continuous from $\mathcal{H}_p \rightarrow \mathbb{C}$. This condition is not always realized and must be checked individually.

Within the canonical quantization there are infinite terms which appear in the energy-momentum tensor and its expectation values for some field states, especially in the vacuum. This calculation rests on the ccr and on the definition of the vacuum $a_k |0\rangle = 0$. In the following section, we describe a new quantization verifying the ccr but with a different vacuum: we will show that these infinite terms actually disappear.

4. Krein space quantization

This quantization is of the Gupta–Bleuler type in the sense that one will distinguish the space on which the observables are defined (called the total space) from the subspace of physical states on which the average values of the observables are computed. Note that the total space is equipped with an indefinite inner product which means that some (unphysical) states have a negative norm.

The original Gupta–Bleuler quantization was invented in order to preserve the (manifest) covariance in the presence of gauge invariance. Not surprisingly, a similar method works in the case of the dS massless minimally coupled scalar field (contrary to the usual canonical quantization) in which gauge invariance also occurs.

We use a set of modes satisfying the conditions given above. The main difference is that we no longer require that the space \mathcal{H}_p is closed under the group action. It is enough that the larger space $\mathcal{H} = \mathcal{H}_p + \mathcal{H}_p^*$ is closed under the latter, which is obviously a weaker condition. Again, the field is defined by equation (8). Now, the representation of the ccr is obtained in the following way. We consider the Fock space $\underline{\mathcal{H}}$ [22] constructed on the Krein space [23]

$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_p^*$, where \mathcal{H}_p^* is the anti-Hilbertian space generated by the set ϕ_k^* . Let us set $a_k = a(\phi_k)$ and $b_k = a(\phi_k^*)$ then

$$A_k = \frac{1}{\sqrt{2}}(a_k - b_k^\dagger) \quad \text{and} \quad A_k^\dagger = \frac{1}{\sqrt{2}}(a_k^\dagger - b_k).$$

Since these operators verify the ccr, we obtain a further realization of these relations which yields a new definition of the field. A significant difference with the usual representation is that the operators A_k do not verify $A_k|0\rangle = 0$, where $|0\rangle$ is the Fock vacuum. We can now point by point summarize the discussion of the above paragraph. The first two conditions (field equation and micro-causality) are still verified since they do not depend on the involved representation of the ccr.

In order to show the covariance, we start by rewriting the field in the smeared form as in the previous section. We define the distribution p taking values in \mathcal{H} in the following way. For any function f , $p(f)$ is the unique element of \mathcal{H} such that

$$\langle p(f), \psi \rangle = (f, \psi) \quad \forall \psi \in \mathcal{H}. \quad (13)$$

The existence of $p(f)$ is subject to a technical condition on \mathcal{H} : the continuity for each f of the map $\psi \mapsto (f, \psi)$, $\mathcal{H} \rightarrow \mathbb{C}$. Formally, $p(f) = \int p(x)f(x) dx$ yields the formal definition $\langle p(x), \psi \rangle = \psi(x)$. It is easy to verify that the kernel \tilde{G} of p is given by

$$\langle p(x), p(x') \rangle = -i\tilde{G}(x, x').$$

Similarly, one can easily check that the field reads

$$\varphi(x) = a(p(x)) + a^\dagger(p(x)). \quad (14)$$

The advantage of the formulation (14) is that the covariance of the field is shown as above (equation (12) and the corresponding demonstration), the only condition being that the space \mathcal{H} (not \mathcal{H}_p) must be closed under the group action. This is a remarkable advantage of the method since, as discussed in detail in [1, 24, 25], situations do arise in which the space \mathcal{H} does not admit a closed invariant Hilbert subspace, and consequently it is not possible to define \mathcal{H}_p in a covariant way.

The total space \mathcal{H} contains negative norm states and cannot be considered as the space of physical states. The latter must be a subspace \mathcal{K} which is not necessarily a Hilbert space but a closed invariant subspace. On \mathcal{K} , the inner product is positive, possibly indefinite. In this case the space $\mathcal{N} = \mathcal{K} \cap \mathcal{K}^\perp$ is the set of unobservable gauge states and completes the so-called Gupta–Bleuler triplet

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{H}.$$

The space \mathcal{K}/\mathcal{N} is the set of physical states in strict sense. This yields a condition for a self-adjoint operator to be an observable: gauge change must not be observed (see [24]). Moreover, when \mathcal{N} is non-trivial, the representation of the group is not irreducible, but only indecomposable. That is the space \mathcal{K} is an invariant subspace of \mathcal{H} but it is not invariantly complemented in \mathcal{H} : there is no invariant subspace \mathcal{L} such that

$$\mathcal{H} = \mathcal{K} \oplus \mathcal{L}.$$

The same scheme occurs for \mathcal{N} in \mathcal{K} . In the usual case (e.g. massive scalar Minkowskian field) the physical space $\mathcal{K} = \mathcal{H}_p$ is invariantly complemented since

$$\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_p^*.$$

Note that this field is, by construction (it depends only on \mathcal{H}), independent of any Bogoliubov transformations which only modify the set of physical states. The space of physical states depends on the considered spacetime, but it also depends on the observer: the

physical states of an accelerated observer in Minkowski space are different from those of an inertial observer (Unruh effect). In our formalism, the same representation of the field can be used for both cases. Even more, following Wald, there is a case where the Krein space appears in a very natural way. ‘For a space-time which is asymptotically stationary in both the past and the future we have two natural choices of vacua, and the S matrix should be a unitary operator between both structures’ [3], p 66. As pointed out by the author, this is not possible if the two vacua are not equivalent. Once again our construction furnishes a stage where all these objects can be considered together. The difference with the algebraic approach is that states (physical or not) are naturally realized as wavefunctions. This is not the case in the algebraic theory where this representation requires a choice which is equivalent to the choice of vacuum.

This method not only allows us to quantize some fields resisting to the usual procedures, but it presents a very interesting property: the vacuum energy vanishes without any reordering and no infinite term appears in the calculation of mean values of $T_{\mu\nu}$. Moreover, this is independent of the curvature. The point is

$$[b_k, b_k^\dagger] = [a(\phi_k^*), a^\dagger(\phi_k^*)] = \langle \phi_k^*, \phi_k^* \rangle = -1.$$

The vacuum is now the Fock vacuum and a straightforward computation leads to

$$\langle 0 | (A_k A_k^\dagger + A_k^\dagger A_k) | 0 \rangle = 0,$$

from which the result follows easily (see [1] for details). This shows that this type of quantization does not require any regularization of the $T_{\mu\nu}$: the average value of the stress tensor in the vacuum state is not only finite, but it actually is zero.

Thus, under weaker condition than in the usual canonical scheme, the Krein space quantization allows us to define a causal and covariant field which features a remarkable automatic regularization mechanism. It has been successfully applied to the case of the massless minimally coupled scalar field in de Sitter space (with the field equation $\square\phi = 0$) [1]. This case is very important because it also appears in the dS massless spin-2 case, that is the graviton on de Sitter space.

The Krein space quantization gives rise to unphysical states. An essential question then arises: how to select the physical states?

First of all, it should be emphasized that in the presence of gauge invariance, the occurrence of Krein structure is unavoidable. Even the standard Gupta–Bleuler quantization involves a Krein space. The one-particle physical state space must be a positive inner product subspace of \mathcal{H} , closed under the group action. Generally, this choice is not unique and we end up with the usual ambiguities which plague quantum fields theories in curved spacetimes. Note however the different standpoints: the vacuum state is considered here as the vacuum state of the Fock space. It is invariant under Bogoliubov transformations which are merely a simple change of the physical state space. It must be emphasized that in the presence of gauge invariance, that space would not be a Hilbert space.

5. An interacting theory in Minkowski spacetime

As indicated in the previous sections, our quantization scheme has already been applied to the dS minimally coupled massless scalar field. In that context it allows us to quantize the free field without losing the covariance. Interesting features are that the vacuum is invariant under a Bogoliubov transformation and no infinite term appears in the computation of $\langle 0 | T^{00} | 0 \rangle$.

However, only free fields have been studied so far. This yields the natural question: how such features behave when interaction is taken into account?

More precisely, let us investigate the way in which the Krein quantization method can be extended to the simple example of an interacting scalar field (called k -scalar field in the following) in Minkowski spacetime defined through the Lagrangian density

$$\mathcal{L} = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi - V(\phi).$$

We will see in the following that, with a standard choice for $V(\phi)$, we recover the usual results of interacting field theory.

The unitarity condition, which we discuss now, takes a different form in our Krein space context. Here, in the absence of gauge invariance, we have $\mathcal{K} = \mathcal{H}_p$. The free field Fock space is denoted by

$$\underline{\mathcal{H}} = \underline{\mathcal{H}_p} \oplus \underline{\mathcal{H}_p^*}.$$

Moreover the space of physical states $\underline{\mathcal{H}_p}$ is closed and positive, hence complementarizable

$$\underline{\mathcal{H}} = \underline{\mathcal{H}_p} \oplus (\underline{\mathcal{H}_p})^\perp.$$

The positive energy (resp. negative energy) one-particle eigenstates of the free Hamiltonian are denoted as $|k\rangle$ (resp. $|\bar{k}\rangle$). The corresponding symmetrized and normalized many-particle states are denoted by $|k_1, \dots, k_m; \bar{k}_1, \dots, \bar{k}_n\rangle$. We shall adopt the abbreviated notation

$$\begin{aligned} |\alpha_+\rangle &\equiv |k_1, \dots, k_m\rangle, \\ |\alpha_-^{(n)}\rangle &\equiv |k_1, \dots, k_m; \bar{k}_1, \dots, \bar{k}_n\rangle, \quad n \neq 0. \end{aligned}$$

A state without sub- or superscripts has an arbitrary number of k and \bar{k} .

If one proceeds to the computation of the S matrix as usual:

$$S = : \exp \left(i \int V(\phi) \right) :, \quad (15)$$

one obtains obviously an operator S which is Krein-unitary. In view of the non-positiveness of the scalar product, the unitarity of the S matrix over a Krein space does not lead to the usual relations

$$\sum_{\{\beta\}} |\langle \beta | S | \alpha \rangle|^2 = 1, \quad (16)$$

$$\sum_{\{\beta\}} |\langle \alpha | S | \beta \rangle|^2 = 1. \quad (17)$$

Indeed, for S such that $S^\dagger S = S S^\dagger = \text{Id}$ (Krein-unitarity) one has

$$\sum_{\{\alpha_+\}} |\langle \alpha_+ | S | \alpha \rangle|^2 + \sum_{\{\alpha_-^{(n)}\}} (-1)^n |\langle \alpha_-^{(n)} | S | \alpha \rangle|^2 = 1, \quad (18)$$

$$\sum_{\{\alpha_+\}} |\langle \alpha | S | \alpha_+ \rangle|^2 + \sum_{\{\alpha_-^{(n)}\}} (-1)^n |\langle \alpha | S | \alpha_-^{(n)} \rangle|^2 = 1. \quad (19)$$

Obviously, equations (18) and (19) cannot share the same usual probabilistic interpretation as equations (16) and (17) and thus the Krein-unitarity has no physical meaning. If one ignores the problem and compute, for $V(\phi) = \lambda \phi^4$ for instance, the S matrix using (15) anyway, one obtains a theory which is identical to the usual theory at tree level but not at higher orders. Precisely, one can show (through straightforward complex integration) that all diagrams containing loops are equal to zero. This is not in contradiction with the optical theorem because the S matrix in this context is not unitary but Krein-unitary. The

above property then reflects the fact that the Krein-unitarity is a different requirement than unitarity. Consequently, the implementation of unitarity in the Krein space context leads to a modification of usual interacting terms.

The sums in equations (18) and (19) extend over the whole Fock space whereas only the states which belong to $\underline{\mathcal{H}}_p$, i.e., with positive norm, are physical states. Thus, only these states can pretend to be the so-called ‘in’ states of the interacting theory. In other words, the space of ‘in’ states cannot be isomorphic to the whole Fock space $\underline{\mathcal{H}}$ but only to its subspace $\underline{\mathcal{H}}_p$. Since in addition the space of ‘out’ states is in one-to-one correspondence with the space of ‘in’ states we conclude that a physically admissible S matrix cannot connect physical states to unphysical ones. Finally the S matrix must satisfy

$$\begin{cases} \sum_{\{\alpha_+\}} |\langle \alpha_+ | S | \alpha \rangle|^2 = \sum_{\{\alpha_+\}} |\langle \alpha | S | \alpha_+ \rangle|^2 = 1 \\ |\langle \alpha_-^{(n)} | S | \alpha \rangle|^2 = |\langle \alpha | S | \alpha_-^{(n)} \rangle|^2 = 0, \end{cases} \quad (20)$$

where $\alpha \in \underline{\mathcal{H}}$.

This can be achieved in the following way. Let Π_+ be the projection over $\underline{\mathcal{H}}_p$:

$$\Pi_+ = \sum_{\{\alpha_+\}} |\alpha_+\rangle \langle \alpha_+|.$$

Indeed equation (8) reads

$$\begin{aligned} \varphi(x) &= \sum_k \{ (a_k \phi_k(x) + a_k^\dagger \phi_k^*(x)) - (b_k^\dagger \phi_k(x) + b_k \phi_k^*(x)) \} \\ &= \varphi_+ - \varphi_-, \end{aligned} \quad (21)$$

and this defines the (φ_+) and (φ_-) parts. Then one has

$$\Pi_+ \varphi \Pi_+ |\gamma\rangle = \begin{cases} \varphi_+ |\gamma\rangle & \text{if } |\gamma\rangle \in \underline{\mathcal{H}}_p \\ 0, & \text{if } |\gamma\rangle \in (\underline{\mathcal{H}}_p)^\perp. \end{cases} \quad (22)$$

Consequently, we start with $V(\Pi_+ \varphi \Pi_+)$ instead of $V(\varphi)$ in deriving the S matrix.

By following this quantization procedure for the interaction term, the k -scalar field theory will lead to the same predictions as the usual scalar field theory. The only difference lies in the vacuum energy density: in the k -scalar field theory the free vacuum energy density is always zero. When interaction is present the vacuum energy density receives (divergent) contributions of higher-order vacuum bubble graphs. Note that the so-called radiative corrections are the same in both theories. In this respect vacuum effects in k -scalar field theory only involve the interacting vacuum. This may appear as an advantage if one believes that a truly free theory is a theoretical fiction.

Note that the new interaction term $V(\Pi_+ \varphi \Pi_+)$ is in some sense the quantum version of the restriction V' of V to the positive energy ϕ 's. In this sense the Lagrangian density should be written as

$$\mathcal{L} = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi - V'(\phi).$$

The Krein quantization of other canonically quantizable theories may be carried out following the same lines. A prescription for interacting terms is to replace the various fields ξ by their restriction $\Pi_+^\xi \xi \Pi_+^\xi$, where Π_+^ξ is the corresponding projector. Such a k -field theory will only differ from the usual one by the vanishing of the free field vacuum energy.

Now, we must underline that, in a curved space, the operator Π_+ may not exist, and the problem of interacting field on such a space is, as is well known, much more elaborated. Note that, in particular, the function $G^{(1)}(x, x') = \langle 0 | \{ \varphi(x), \varphi(x') \} | 0 \rangle$ vanishes in our formalism,

and thus cannot have the Hadamard property. This could be a problem for the construction of perturbative interacting field theories in curved space [26, 27]. Nevertheless, the link between the vacuum and the two-point function is not exactly the same as in the usual quantization scheme and in the Krein space quantization context: in the usual approach to choose a vacuum is to choose a physical space of states and a two-point function; in our approach the vacuum is unique and consequently do not determine the physical space of states but the link between this space and the two-point function remains [1]. As a consequence a two-point function with Hadamard property is still present but with another meaning. The description of an interacting theory using the Krein-space quantization approach remains open.

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Appendix. The fock space

Any Krein space \mathcal{H} is equipped with a (non-unique) Hilbert space structure. Hence, one can define over \mathcal{H} a Fock space through [22]

$$\underline{\mathcal{H}} = \bigoplus_{n \geq 0} S_n(\mathcal{H}),$$

where $S_n(\mathcal{H})$ is the n th symmetric tensor product \mathcal{H} . In a Fock space creation and annihilation operators are well defined not only for the modes but also for any element of \mathcal{H} . Suppose that the scalar product is defined through (3); then one has

$$(a(\phi)\Psi)(x_1, \dots, x_{n-1}) = \sqrt{n} \int_{\Sigma} \phi^*(x) \overleftrightarrow{\partial}_{\mu} \Psi(x, x_1, \dots, x_{n-1}) d\sigma^{\mu}(x),$$

for any $\Psi \in S_n(\underline{\mathcal{H}})$. The creation operator reads

$$(a^{\dagger}(\phi)\Psi)(x_1, \dots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \phi(x_i) \Psi(x_1, \dots, \check{x}_i, \dots, x_{n+1}).$$

One verifies that

$$[a(\phi), a(\phi')] = 0 = [a^{\dagger}(\phi), a^{\dagger}(\phi')] = 0$$

$$[a(\phi), a^{\dagger}(\phi')] = \langle \phi, \phi' \rangle,$$

and

$$\underline{U}_g a^{\dagger}(\phi) \underline{U}_g^* = a^{\dagger}(U_g \phi) \quad \text{and} \quad \underline{U}_g a(\phi) \underline{U}_g^* = a(U_g \phi), \quad (\text{A1})$$

where U is the natural representation of the group G on \mathcal{H} and \underline{U} is the extension of this representation to the Fock space.

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